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A Characterization of the T_m Graph*

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ABSTRACT

Let n and m be positive integers with $n \geqslant 2m$. The T_m graph with characteristic n, denoted by G_m^n, is defined as a graph for which the vertices may be identified with all unordered m-tuples on n symbols so that two vertices are adjacent if and only if the corresponding m-tuples contain a common $(m-1)$-tuple. The present paper establishes a characterization of G_m^n when $n > 2m(m-1)+4$ in terms of conditions similar to, but slightly weaker than, the conditions used by Connor [3] to characterize the triangular (T_2) graph and by Bose and Laskar [2] to characterize the tetrahedral (T_3) graph.

1. BASIC TERMINOLOGY

By a *graph* we shall mean a finite, undirected graph with no loops and no multiple edges. A graph G consists of a finite, non-empty set of *vertices* $V(G)$ and a set of *edges* $E(G)$, each of which is an unordered pair of distinct vertices. The edge (u, v) is *incident* with each of its vertices, and two vertices joined by an edge are *adjacent*. A graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

A *chain* $C = (u_0, u_1, ..., u_t)$ *of length* t is a sequence of $t + 1 \geqslant 2$ vertices for which any two consecutive vertices in the sequence are adjacent. Thus $(u_i, u_{i+1}) \in E(G)$ for $i = 0, 1, ..., t-1$. C is said to *join* u_0 and u_t.

If there exists a chain joining u and v for every distinct pair $u, v \in V(G)$, then G is a *connected graph*. Every graph G can be uniquely partitioned into $s \geqslant 1$ subgraphs $G_1, G_2, ..., G_s$, such that each G_i is connected and no edge joins a vertex of G_i to a vertex of $G_j, j \neq i$. The subgraphs $G_1, G_2, ..., G_s$ are the *connected components* of G. For any two vertices u, v in the same connected component of G, the *distance* $d(u, v)$ between

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u and v is the length of the shortest chain joining u and v. Clearly $d(u, v) = 1$ if and only if $(u, v) \in E(G)$. By convention we take $d(u, u) = 0$ for all $u \in V(G)$.

The degree of a vertex u, written $\deg u$, is the number of edges incident at u, i.e., the number of vertices adjacent to u. If every vertex has the same degree d, then G is a *regular* graph with degree d.

Following Bose and Laskar [2], we define $\Delta(u, v)$ as the number of vertices adjacent to both u and v. Clearly $\Delta(u, v) = 0$ if $d(u, v) > 2$ or if u and v belong to separate connected components of G. If $(u, v) \in E(G)$, $\Delta(u, v)$ is called the *edge-degree* of the edge (u, v). A regular graph G for which all edges have the same edge-degree Δ is said to be *edge-regular* with edge-degree Δ.

A *clique* K is a set of vertices, any two of which are adjacent. K is *complete* if, for any vertex $u \notin K$, the set $K \cup u$ is not a clique.

If V_0 and V_1 are two disjoint sets of vertices in $V(G)$, we define a *bridge joining* V_0 *and* V_1 as an edge $(v_0, v_1) \in E(G)$, where $v_i \in V_i$, $i = 0, 1$. When there is no ambiguity as to the sets V_0 and V_1, a bridge joining V_0 and V_1 will be called simply a *bridge*.

The cardinality of a set U will be denoted by $|U|$. The set consisting of the elements $u_1, u_2, \ldots, u_n$ will be denoted by $(u_1, u_2, \ldots, u_n)$. We shall denote by u both the element u and the set consisting of the single element u.

2. Statement of Results

Let n and m be positive integers with $n \geqslant 2m$. A *T_m graph with characteristic n*, denoted by G_m^n, is defined as a graph for which the vertices can be identified with all unordered m-tuples on n symbols so that two vertices are adjacent if and only if the corresponding m-tuples contain a common $(m-1)$-tuple. It is easily verified that, if $G = G_m^n$, then G is a connected graph with the following properties:

(a_0) $\quad |V(G)| = \binom{n}{m}$,

(a_1) $\quad \deg u = m(n-m)$ for all $u \in V(G)$,

(a_2) $\quad \Delta(u, v) = n - 2$ if $(u, v) \in E(G)$,

(a_3) $\quad \Delta(u, v) \leqslant 4$ if $(u, v) \notin E(G)$.

If we regard a 0-tuple as the empty set, then it is clear that G_1^n is the *complete graph* on n vertices, i.e., the graph G for which $(u, v) \in E(G)$ for every distinct pair $u, v \in V(G)$. It is trivially true that, if G satisfies $(a_0) - (a_3)$ with $m = 1$, then $G = G_1^n$.

Using the terminology of association schemes for partially balanced designs, Connor [3] showed that, if $m = 2$ and equality holds in (a_3) when $d(u, v) = 2$, then properties (a_0)–(a_3) characterize the triangular or T_2 graph if $n > 8$; i.e., if G satisfies (a_0)–(a_3) with $m = 2$ and $n > 8$, then $G = G_2^n$. Shrikhande [8], Chang Li-Chien [6, 7], and Hoffman [4, 5] completed the characterization of the triangular graph by showing that the same result holds when $n < 8$, but that if $n = 8$ there exist exactly three non-isomorphic graphs which satisfy (a_0)–(a_3) (assuming equality in (a_3) when $d(u, v) = 2$), but which are not triangular.

More recently Bose and Laskar [2] considered the tetrahedral or T_3 graph, and showed[1] that, again assuming $\Delta(u, v) = 4$ if $d(u, v) = 2$, properties (a_0)–(a_3) with $m = 3$ characterize the tetrahedral graph if $n > 16$. Aigner [1] has obtained the analogous result for $n \leqslant 8$.

In the present paper we generalize the results of [2] and [3] to an arbitrary $m \geqslant 2$, without assuming equality in (a_3) when $d(u, v) = 2$. We also show that the same result holds if (a_0) is replaced by the condition that G be connected. This latter result, Theorem 1, is stated below and proved in Section 3. We also state and prove, using Theorem 1, two immediate corollaries.

THEOREM 1. *If G is a connected graph satisfying* (a_1)–(a_3) *for $n > 2m(m - 1) + 4$, then $G = G_m^n$.*

COROLLARY 1. *If G is a graph satisfying* (a_1)-(a_3) *for $n > 2m(m - 1) + 4$, then $G_0 = G_m^n$ for every connected component G_0 of G.*

PROOF: G_0 is connected and also satisfies (a_1)–(a_3).

COROLLARY 2. *If G is a graph satisfying* (a_0)-(a_3) *for $n > 2m(m - 1) + 4$, then $G = G_m^n$.*

PROOF: If G_0 is a connected component of G, then $G_0 = G_m^n$ by Corollary 1. Hence $|V(G_0)| = \binom{m}{n}$, i.e., G is connected, so Theorem 1 applies.

REMARK. Corollary 2 is the generalization of the earlier characterizations of the triangular graph [3] and tetrahedral graph [2], although our condition (a_3) is slightly weaker. Actually Corollary 2 could be proved directly with minor changes in the proof of Theorem 1 below. It is more instructive, however, to derive it as an immediate consequence of Theorem 1. In so doing, we see that (a_0) is needed only to ensure that G is not the

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[1] In [2] it is assumed additionally that G is connected, but this property is not used and the arguments remain valid when it is dropped.

disjoint sum of two or more T_m graphs. Note in particular that Corollary 2 remains valid if we replace (a_0) by the weaker condition

$$| V(G)| < 2\binom{n}{m}.$$

It is natural to ask what happens when $n \leqslant 2m(m-1)+4$. As noted above, (a_0)–(a_3) with $m = 2$ characterize G_2^n for all $n \neq 8$. At present the author is unaware of the existence of any exceptional cases when $m \geqslant 3$.

3. Proof of Theorem 1

We shall first state for completeness a theorem on edge-regular graphs due to Bose and Laskar [2] which is fundamental to the proof of Theorem 1.

THEOREM (Bose-Laskar). *Let G be a graph satisfying the following conditions*:

(c_1) $\deg u = r(k-1)$ *for all* $u \in V(G)$,

(c_2) $\Delta(u, v) = k - 2 + \alpha$ *if* $(u, v) \in E(G)$,

(c_3) $\Delta(u, v) \leqslant 1 + \beta$ *if* $(u, v) \notin E(G)$,

where $r \geqslant 1$, $k \geqslant 2$, $\alpha \geqslant 0$, $\beta \geqslant 0$ *are fixed integers. Define a grand clique K as a complete clique with* $|K| \geqslant k - (r-1)\alpha$. *If*

$$k > \max[p(r, \alpha, \beta), \rho(r, \alpha, \beta)],$$

where

$$p(r, \alpha, \beta) = 1 + \tfrac{1}{2}(r+1)(r\beta - 2\alpha),$$
$$\rho(r, \alpha, \beta) = 1 + \beta + (2r-1)\alpha,$$

then

(i) *each vertex of G is contained in exactly r grand cliques*;

(ii) *each pair of adjacent vertices is contained in exactly one grand clique.*

REMARK. A graph G satisfying (i) and (ii) has a type of geometric structure which will be exploited in the proof of Theorem 1. We may identify the vertices with points and the grand cliques with lines in this "geometry," and the theorem then states that r lines pass through each point and at most one line passes through any pair of points, such a line existing if and only if the corresponding vertices are adjacent in G. It follows that any two lines intersect in at most one point. We note also that the $r(k-1)$ vertices adjacent to a given vertex u correspond to the points other than u on the r lines containing u.

The proof of Theorem 1 will be by induction on m for fixed n. In the case $m = 1$, the conditions of the theorem state that G is a connected, edge-regular graph of degree $n - 1$ and edge-degree $n - 2$. Thus G is obviously the complete graph on n vertices, i.e., $G = G_1^n$, so that the theorem holds for $m = 1$.

In the remainder of this section, we shall assume $m \geqslant 2$. We note first that conditions (a_1)–(a_3) are equivalent to conditions (c_1)–(c_3) of the Bose-Laskar theorem with $r = m$, $k = n - m + 1$, $\alpha = m - 1$, $\beta = 3$. Thus a grand clique K in G is defined as a complete clique such that

$$| K | \geqslant (n - m + 1) - (m - 1)^2 = n - m(m - 1). \tag{1}$$

Since $m \geqslant 2$ and $n > 2m(m - 1) + 4$, every grand clique contains at least seven vertices. We also have

$$p(m, m-1, 3) = 1 + \tfrac{1}{2}(m + 1)(m + 2),$$
$$\rho(m, m-1, 3) = 4 + (2m - 1)(m - 1),$$

and thus

$$\rho(m, m-1, 3) - p(m, m-1, 3) = \tfrac{3}{2}(m - 1)(m - 2) \geqslant 0.$$

Hence the Bose-Laskar theorem holds if

$$n - m + 1 > 4 + (2m - 1)(m - 1),$$

i.e., if

$$n > 2m(m - 1) + 4,$$

a condition we have assumed. We therefore have

LEMMA 1. *Each vertex of G is contained in exactly m grand cliques, and each pair of adjacent vertices is contained in exactly one grand clique.*

Denote by $\mathscr{C}(G)$ the set of all grand cliques in G, and let $\chi(G)$ be the set of all unordered pairs (K_0, K_1) of grand cliques such that $K_0 \cap K_1 \neq \phi$. It follows from Lemma 1 that, if $(K_0, K_1) \in \chi(G)$, then K_0 and K_1 intersect in a unique vertex u, which we write as $u = K_0 \cap K_1$.

Let $(K_0, K_1) \in \chi(G)$, and let $u = K_0 \cap K_1$. We define $B(K_0, K_1)$ as the total number of bridges joining $K_0 - u$ and $K_1 - u$. We then have

LEMMA 2. *If $(K_0, K_1) \in \chi(G)$, then*

$$B(K_0, K_1) \leqslant \max(| K_0 |, | K_1 |) + 1.$$

PROOF: Let $u = K_0 \cap K_1$ and define b as the maximum number of bridges incident at a vertex of $(K_0 - u) \cup (K_1 - u)$. The lemma is trivially true if $b = 0$, so assume $b \geqslant 1$ and let u_0 be a vertex incident with b bridges. Assume with no loss of generality that $u_0 \in K_0 - u$, and let $v_1, v_2, \ldots, v_b$ be the vertices of $K_1 - u$ adjacent to u_0. It is clear by Lemma 1 that each of the vertices $u, v_1, v_2, \ldots, v_b$ must lie in a different grand clique containing u_0, and thus we have $b + 1 \leqslant m$. Since K_1 is a grand clique, and is therefore complete, there exists a vertex $v \in K_1$ not adjacent to u_0. Clearly $u, v_1, v_2, \ldots, v_b$ are $b + 1$ vertices adjacent to both u_0 and v, so that (a_3) implies $b \leqslant 3$. Again by (a_3) and the definition of b, the number of vertices in $K_0 - u$ adjacent to v cannot exceed $\min(b, 3 - b)$. Since the number of vertices in $K_1 - u$ not adjacent to u_0 is $|K_1| - b - 1$, and since each of $v_1, v_2, \ldots, v_b$ is adjacent to at most b vertices in $K_0 - u$, including u_0, we have

$$B(K_0, K_1) \leqslant b^2 + (|K_1| - b - 1)\min(b, 3 - b), \tag{2}$$

where $b \leqslant 3$ and $b + 1 \leqslant m$.

If we denote the right-hand side of (2) by $\varphi(b)$, then $\varphi(3) = 9$, $\varphi(2) = |K_1| + 1$, $\varphi(1) = |K_1| - 1$. Using (1) and the condition $n > 2m(m - 1) + 4$, it is easily vertified that $\varphi(2) \geqslant \varphi(3)$ when $m \geqslant 3$. Hence, if we assume that a vertex u_0 incident with b bridges is in $K_0 - u$, we obtain $B(K_0, K_1) \leqslant |K_1| + 1$. By symmetry, if $u_0 \in K_1 - u$, we obtain $B(K_0, K_1) \leqslant |K_0| + 1$, so that in any case the lemma holds.

LEMMA 3. *If $K \in \mathscr{C}(G)$, then*

$$|K| = n - m + 1.$$

PROOF: Let u be an arbitrary vertex and let $K_0, K_1, \ldots, K_{m-1}$ be the m grand cliques containing u. Let $N_i = |K_i|$, $i = 0, 1, \ldots, m - 1$, and assume without loss of generality that

$$N_0 = \min N_i, \qquad i = 0, 1, \ldots, m - 1. \tag{3}$$

By Lemma 1, a vertex v is adjacent to u if and only if $v \in K_i$ for exactly one i, $i = 0, 1, \ldots, m - 1$. Hence by (a_1), we have

$$\sum_{i=0}^{m-1} N_i = m(n - m + 1). \tag{4}$$

It follows from (3) and (4) that

$$N_0 \leqslant n - m + 1. \tag{5}$$

Let

$$L = \bigcup_{i=1}^{m-1} K_i$$

and consider the total number of bridges joining $L - u$ and $K_0 - u$. If $u_0 \in K_0 - u$, there are $N_0 - 2$ vertices in K_0 adjacent to both u and u_0. Since $\Delta(u, u_0) = n - 2$ by (a$_2$), there remain $n - N_0$ vertices adjacent to both u and u_0 and not in K_0, and these must clearly lie in $L - u$. Since u_0 can be chosen in $N_0 - 1$ ways, the total number of bridges joining $K_0 - u$ and $L - u$ is $(n - N_0)(N_0 - 1)$. Counting these edges in another way and using Lemma 2, (3), and (4), we have

$$\begin{aligned}(n - N_0)(N_0 - 1) &= \sum_{i=1}^{m-1} B(K_0, K_i)\\ &\leqslant \sum_{i=1}^{m-1} (N_i + 1)\\ &= [m(n - m + 1) - N_0] + (m - 1).\end{aligned}$$

On simplification, this inequality becomes

$$N_0^2 - (n + 2)\, N_0 + [(m + 1)(n - m + 1) + 2(m - 1)] \geqslant 0. \tag{6}$$

The discriminant of the quadatic $f(N_0)$ on the left-hand side of (6) is

$$\begin{aligned}D^2 &= (n + 2)^2 - 4[(m + 1)(n - m + 1) + 2(m - 1)]\\ &= (n - 2m - 2)^2 + 4(n - 4m + 1).\end{aligned}$$

Since $n > 2m(m - 1) + 4$, $n - 4m + 1 > 0$ and it follows that the roots of $f(N_0)$ are real and that $D > n - 2m - 2$. Hence (6) implies

$$| N_0 - \tfrac{1}{2}(n + 2)| \geqslant \tfrac{1}{2}D.$$

Using (1) and the condition $n > 2m(m - 1) + 4$, it is easily verified that $N_0 > \frac{1}{2}(n + 2)$. Hence

$$\begin{aligned}N_0 &\geqslant \tfrac{1}{2}(n + 2 + D)\\ &> \tfrac{1}{2}(n + 2 + n - 2m - 2)\\ &= n - m.\end{aligned}$$

Thus by (5) we have $N_0 = n - m + 1$, and (3) and (4) therefore imply that $N_i = n - m + 1$ for $i = 0, 1,..., m - 1$. Since u is an arbitrary vertex and every grand clique is non-empty, it follows that $|K| = n - m + 1$ for all $K \in \mathscr{C}(G)$.

COROLLARY 3. *If* $(K_0, K_1) \in \chi(G)$, *then*

$$B(K_0, K_1) \leqslant n - m + 2.$$

LEMMA 4. *If* $(K_0, K_1) \in \chi(G)$, *then*

$$B(K_0, K_1) \geqslant n - m^2 + 2.$$

PROOF: Let $u = K_0 \cap K_1$ and let $K_2, K_3, \ldots, K_{m-1}$ be the remaining grand cliques containing u. We shall count the total number of bridges joining $K_i - u$ and $K_j - u$ for all distinct pairs (i, j), $i, j = 0, 1, \ldots, m - 1$. If v is one of the $m(n - m)$ vertices adjacent to u, then $v \in K_i - u$ for some i. By (a_2) and Lemma 3 there are $m - 1$ vertices adjacent to both u and v and not in K_i. Corresponding to each of these is a bridge joining $K_i - u$ and $K_j - u$ for some $j \neq i$. Since on taking all $m(n - m)$ choices for v, each bridge is counted twice, the required number of bridges is $m(m - 1)(n - m)/2$. Counting these in another way, we have

$$\sum_{0 \leqslant i < j \leqslant m-1} B(K_i, K_j) = \binom{m}{2} (n - m). \tag{7}$$

If for all $(i, j) \neq (0, 1)$ we replace $B(K_i, K_j)$ in (7) by the upper bound of Corollary 3, we then obtain

$$\begin{aligned} B(K_0, K_1) &\geqslant \binom{m}{2} (n - m) - \left[\binom{m}{2} - 1\right] (n - m + 2) \\ &= n - m^2 + 2. \end{aligned}$$

LEMMA 5. *If* $K \in \mathscr{C}(G)$ *and* $u \notin K$, *then there are at most two vertices in* K *adjacent to* u.

PROOF: Define an integer t as the maximum number of vertices in K adjacent to u, where the maximum is taken over all pairs u, K such that $K \in \mathscr{C}(G)$, $u \notin K$. The lemma is trivially true if $t = 0$ or $m = 2$, so assume $t \geqslant 1$ and $m \geqslant 3$. Let u, K be a pair for which this maximum is attained. Let $v_0, v_1, \ldots, v_{t-1}$ be the vertices in K adjacent to u, and let K_i, $i = 0, 1, \ldots, t - 1$ be the grand clique containing u and v_i. Since K is complete there exists $v \in K$ such that $(u, v) \notin E(G)$, and hence by (a_3) it is clear that $t \leqslant 4$. Define $K^* = K - (v_0, v_1, \ldots, v_{t-1})$ and $K_i^* = K_i - (u, v_i)$, $i = 0, 1, \ldots, t - 1$. Consider the total number of bridges joining K^* and L^*, where

$$L^* = \bigcup_{i=0}^{t-1} K_i^*.$$

If $v \in K^*$, there are at most $4 - t$ vertices in L^* adjacent to both u and v. Since $|K^*| = n - m + 1 - t$, the total number of bridges joining K^* and L^* is bounded above by $(4 - t)(n - m + 1 - t)$. Hence for at least one K_i^*, say K_0^*, the number of bridges joining K^* and K_0^* is at most $(4 - t)(n - m + 1 - t)/t$. The total number $B(K, K_0)$ of bridges joining $K - v_0$ and $K_0 - v_0$ consists of these plus the bridges joining $K_0 - v_0$ and $(v_1, v_2, ..., v_{t-1})$. Since each v_i, $i = 1, 2, ..., t - 1$, is adjacent to at most $t - 1$ vertices in $K_0 - v_0$, including u, by the definition of t, we have

$$B(K, K_0) \leqslant (t - 1)^2 + (4 - t)(n - m + 1 - t)/t. \tag{8}$$

If $t = 4$, then (8) implies $B(K, K_0) \leqslant 9$, which contradicts Lemma 4 since $m \geqslant 3$ and $n > 2m(m - 1) + 4$. If $t = 3$, then (8) implies $B(K, K_0) \leqslant (n - m + 10)/3$, which again contradicts Lemma 4 since $n > 2m(m - 1) + 4$. Hence $t \leqslant 2$.

COROLLARY 4. *If* $(K_0, K_1) \in \chi(G)$, *then*

$$B(K_0, K_1) \leqslant n - m.$$

PROOF: Let $u = K_0 \cap K_1$. By Lemma 5 no vertex in $K_0 - u$ can be adjacent to more than one vertex in $K_1 - u$. Hence

$$B(K_0, K_1) \leqslant |K_0 - u| = n - m.$$

LEMMA 6. *If* $(K_0, K_1) \in \chi(G)$, *then*

$$B(K_0, K_1) = n - m,$$

and the vertices of $K_0 - u$ *are in one-one correspondence with the vertices of* $K_1 - u$ *such that* $u_0 \in K_0 - u$ *and* $u_1 \in K_1 - u$ *correspond if and only if* $(u_0, u_1) \in E(G)$.

PROOF: To prove the first part we proceed exactly as in the proof of Lemma 4 except that, for all $(i, j) \neq (0, 1)$, we replace $B(K_i, K_j)$ in (7) by the upper bound of Corollary 4. We then obtain

$$\begin{aligned} B(K_0, K_1) &\geqslant \binom{m}{2}(n - m) - \left[\binom{m}{2} - 1\right])(n - m) \\ &= n - m, \end{aligned}$$

which by Corollary 4 implies that $B(K_0, K_1) = n - m$.

If $u_0 \in K_0 - u$, then by Lemma 5 there is at most one vertex $u_1 \in K_1 - u$ adjacent to u_0. If no such u_1 exists, then $B(K_0, K_1) < n - m$, which contradicts the first part. Hence u_1 exists and is unique. Similarly, given $u_1 \in K_0 - u$ there exists a unique vertex $u_0 \in K_0 - u$ adjacent to u_1.

LEMMA 7. *If $K \in \mathscr{C}(G)$, there are exactly $(m-1)(n-m+1)$ grand cliques which intersect K.*

PROOF: Each of the $n - m + 1$ vertices in K is contained in exactly $m - 1$ other grand cliques, and no two of these $(m-1)(n-m+1)$ grand cliques can be identical by Lemma 1.

LEMMA 8. *If $(K_0, K_1) \in \chi(G)$, there are exactly $n - 2$ grand cliques which intersect both K_0 and K_1.*

PROOF: Let $u = K_0 \cap K_1$. A bridge (u_0, u_1) joining $K_0 - u$ and $K_1 - u$ determines a unique grand clique intersecting both $K_0 - u$ and $K_1 - u$, and conversely, every such grand clique determines a unique bridge (u_0, u_1). Since the number of bridges joining $K_0 - u$ and $K_1 - u$ is $n - m$ by Lemma 6, and since there are $m - 2$ grand cliques other than K_0 and K_1 which contain u, the total number of grand cliques intersecting both K_0 and K_1 is $(n-m) + (m-2) = n - 2$.

LEMMA 9. *If $(K_0, K_1) \notin \chi(G)$, there are at most four grand cliques which intersect both K_0 and K_1.*

PROOF: The lemma is trivially true if no grand clique intersects both K_0 and K_1, so assume there exists at least one such grand clique L_1. Let $u_0 = K_0 \cap L_1$, $u_1 = K_1 \cap L_1$. By Lemma 6 there exist vertices $v_0 \in K_0 - u_0$, $v_1 \in K_1 - u_1$ such that (u_0, v_1), $(u_1, v_0) \in E(G)$. Since $(K_0, K_1) \notin \chi(G)$, $v_0 \neq v_1$. Let L_2, L_3 be the grand cliques containing the pairs (u_0, v_1), (u_1, v_0), respectively. If $(v_0, v_1) \notin E(G)$, then again by Lemma 6 there exist vertices $w_0 \in K_0 - u_0$, $w_1 \in K_1 - u_1$, $w_2 \in L_2 - u_0$ such that (v_0, w_1), (v_1, w_0), $(v_0, w_2) \in E(G)$. Then u_0, u_1, w_0, w_1, w_2 are five vertices adjacent to both v_0 and v_1, which contradicts (a_3). Hence $(v_0, v_1) \in E(G)$, and there exists a grand clique L_4 containing v_0 and v_1.

Suppose there is a fifth grand clique L_5 intersecting both K_0 and K_1. By Lemma 5, L_5 cannot contain any of the vertices u_0, u_1, v_0, v_1. Let $w_0 = K_0 \cap L_5$, $w_1 = K_1 \cap L_5$. Then, by Lemma 6, there exist vertices $x_1 \in L_1 - u_1$, $x_2 \in L_2 - v_1$ such that (w_1, x_1), $(w_1, x_2) \in E(G)$. But then u_1, v_1, w_0, x_1, x_2 are five vertices adjacent to both u_0 and w_1, which contradicts (a_3) since $(u_0, w_1) \notin E(G)$. Hence L_5 cannot exist.

We have shown that Theorem 1 holds in the case $m = 1$. Assume now that $m \geqslant 2$ and the theorem holds for $m^* = m - 1$. Let G^* be the graph whose vertices correspond to the grand cliques of G, and such that two vertices of G^* are adjacent if and only if the corresponding grand cliques intersect. Thus if we denote the correspondence between $u^* \in V(G^*)$ and $K \in \mathscr{C}(G)$ by $u^* \simeq K$, then $E(G^*)$ corresponds to $\chi(G)$ in the sense that if $u_i^* \in V(G^*)$, $K_i \in \mathscr{C}(G)$, and $u_i^* \simeq K_i$, $i = 0, 1$, then $(u_0^*, u_1^*) \in E(G^*)$ if and only if $(K_0, K_1) \in \chi(G)$. Since G is connected, it is clear from Lemma 1 that G^* is also connected. By Lemmas 7–9 we have

(a_1^*) $\deg u^* = m^*(n - m^*)$ for all $u^* \in V(G^*)$,

(a_2^*) $\Delta(u^*, v^*) = n - 2$ if $(u^*, v^*) \in E(G^*)$,

(a_3^*) $\Delta(u^*, v^*) \leqslant 4$ if $(u^*, v^*) \notin E(G^*)$.

These are simply the conditions of the theorem with m^* replacing m. Since $n > 2m(m-1) + 4 > 2m^*(m^* - 1) + 4$, by the inductive hypothesis the theorem holds for G^*, i.e., $G^* = G_{m-1}^n$. In particular this implies

$$| \mathscr{C}(G)| = \binom{n}{m-1}.$$

Since each vertex of G is contained in m grand cliques, and each grand clique contains $n - m + 1$ vertices, the number of vertices in G is

$$| V(G)| = (n - m + 1)\binom{n}{m-1}\Big/m = \binom{n}{m}.$$

The proof will be complete when it is shown that these vertices can be identified with all m-tuples on n symbols so that two vertices are adjacent if and only if the corresponding m-tuples contain a common $(m-1)$-tuple. Now, since $G^* = G_{m-1}^n$, we can identify the grand cliques of G with $(m-1)$-tuples on n symbols so that two grand cliques intersect if and only if the corresponding $(m-1)$-tuples contain a common $(m-2)$-tuple. For $K \in \mathscr{C}(G)$, denote by $C_{m-1}(K)$ the $(m-1)$-tuple corresponding to K. Then if $(K_0, K_1) \in \chi(G)$, and $C_{m-2} = C_{m-1}(K_0) \cap C_{m-1}(K_1)$, C_{m-2} is an $(m-2)$-tuple contained in both $C_{m-1}(K_0)$ and $C_{m-1}(K_1)$. To the vertex $u = K_0 \cap K_1$ we assign the m-tuple

$$C_m(u) = C_{m-1}(K_0) \cup C_{m-1}(K_1). \tag{9}$$

If $m = 2$, then it is clear that the m-tuple $C_m(u)$ assigned to u is unique. If $m \geqslant 3$ we must show that this definition is invariant under any of the $\binom{m}{2}$ possible choices of two grand cliques intersecting in u. For this let K_2 be a third grand clique containing u. If $C_m(u)$ is defined by (9) and

$C_{m-1}(K_2) \not\subset C_m(u)$, then clearly $C_{m-1}(K_2) \supset C_{m-2}$ since K_2 intersects K_0 and K_1. The number of $(m-1)$-tuples containing the $(m-2)$-tuple C_{m-2} is $n - m + 2$, and to each of these corresponds a grand clique in G. There are only m grand cliques containing u and $n - m + 2 > m$ since $n > 2m(m-1) + 4$, and thus there exists $K \in \mathscr{C}(G)$ such that $C_{m-2} \subset C_{m-1}(K)$ but $u \notin K$. Then K intersects each of K_0, K_1, K_2 and therefore contains three vertices adjacent to u. Since $u \notin K$, this contradicts Lemma 5. It follows that $C_{m-1}(K) \subset C_m(u)$. Thus, if $K_0, K_1, \ldots, K_{m-1}$ are the m grand cliques containing u, then $C_{m-1}(K_i) \subset C_m(u)$ for $i = 0, 1, \ldots, m-1$, where $C_m(u)$ is defined by (9). Thus $C_m(u) = C_{m-1}(K_i) \cup C_{m-1}(K_j)$ for any distinct pair (i, j), $i, j = 0, 1, \ldots, m-1$, i.e., $C_m(u)$ is defined uniquely. Since the number of $(m-1)$-tuples contained in $C_m(u)$ is exactly m, we have additionally that

$$C_{m-1}(K) \subset C_m(u) \qquad \text{if and only if} \quad u \in K \tag{10}$$

for any $u \in V(G)$, $K \in \mathscr{C}(G)$.

Let $(u, v) \in E(G)$, and let K be the grand clique containing u and v. Then by (10) we have $C_{m-1}(K) \subset C_m(u)$, $C_{m-1}(K) \subset C_m(v)$. Hence the m-tuples $C_m(u)$ and $C_m(v)$ contain a common $(m-1)$-tuple $C_{m-1}(K)$. Conversely, if u and v are vertices such that $C_m(u)$ and $C_m(v)$ contain a common $(m-1)$-tuple C_{m-1}, then, if K is the grand clique defined by $C_{m-1}(K) = C_{m-1}$, it follows from (10) that $u \in K$, $v \in K$. Hence $(u, v) \in E(G)$. Thus $G = G_m^n$ and the proof is complete.

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